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Journal of Approximation Theory 131 (2004) 59–73
Journal of Approximation Theory
www.elsevier.com/locate/jat

New polynomial preserving operators on simplices: direct results

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Received 30 September 2003; received in revised form 21 April 2004; accepted in revised form 10 September 2004

Communicated by Dany Leviatan

Abstract

A new class of differential operators on the simplex is introduced, which define weighted Sobolev norms and whose eigenfunctions are orthogonal polynomials with respect to Jacobi weights. These operators appear naturally in the study of quasi-interpolants which are intermediate between Bernstein–Durrmeyer operators and orthogonal projections on polynomial subspaces. The quasi-interpolants satisfy a Voronovskaja-type identity and a Jackson–Favard-type error estimate. These and further properties follow from a spectral analysis of the differential operators. The results are based on a pointwise orthogonality relation of Bernstein polynomials that was recently discovered by the authors.

Keywords: Bernstein polynomials; Bernstein–Durrmeyer operators; Quasi-interpolation; Voronovskaja-type theorems; Jackson–Favard-type estimate; Simplex; Orthogonal polynomials

1. Introduction

We denote the barycentric coordinates on the standard simplex

$$S^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \leqslant x_1, \dots, x_d \leqslant 1,\ x_1 + \cdots + x_d \leqslant 1\}$$

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doi:10.1016/j.jat.2004.09.004

by

$$\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_d) = (1 - x_1 - \cdots - x_d, x_1, \dots, x_d).$$

For any multi-index $\alpha = (\alpha_0, \dots, \alpha_d) \in \mathbb{N}_0^{d+1}$, the d-variate basic Bernstein polynomial B_α, of total degree $n := |\alpha|$, is defined by

$$B_\alpha(x_1, \dots, x_d) := \binom{n}{\alpha} \boldsymbol{\lambda}^\alpha = \frac{n!}{\alpha_0! \cdots \alpha_d!} \lambda_0^{\alpha_0} \cdots \lambda_d^{\alpha_d}. \tag{1.1}$$

We will use standard multi-index notation and define $|\alpha| := \alpha_0 + \cdots + \alpha_d$ (without taking absolute values, even if $\alpha \in \mathbf{Z}^{d+1}$) and $\binom{n}{\alpha} := 0$ if any of the components α_i is negative. Under this condition we also set $B_\alpha(x_1, \dots, x_d) := 0$ for all $\mathbf{x} = (x_1, \dots, x_d)$.

The d-simplex S^d has $R := \frac{d(d+1)}{2}$ edges which we denote by

$$e_{ij} := e_j - e_i, \qquad 0 \leqslant i < j \leqslant d,$$

where $e_0 := 0$ and e_i, $1 \leqslant i \leqslant d$, are the unit coordinate vectors. The directional derivatives

$$D_{ij} := \frac{\partial}{\partial e_{ij}} = e_{ij} \cdot \nabla$$

along the edges of S^d constitute the row vector

$$\mathbf{D} := (D_{ij};\ 0 \leqslant i < j \leqslant d),$$

where we choose the lexicographical ordering on the index set $\{0 \leqslant i < j \leqslant d\}$. Furthermore, we define a row vector Λ of quadratic polynomials by

$$\Lambda := (\lambda_i \lambda_j;\ 0 \leqslant i < j \leqslant d)$$

with the same lexicographical ordering of its entries. Again we use standard multi-index notation for $\mathbf{k} = (k_{ij})_{0 \leqslant i < j \leqslant d} \in \mathbb{N}_0^R$, in order to define the differential operators

$$\mathbf{D}^\mathbf{k} := \prod_{0 \leqslant i < j \leqslant d} D_{ij}^{k_{ij}}$$

of order $|\mathbf{k}|$ and the polynomials

$$\Lambda^\mathbf{k} := \prod_{0 \leqslant i < j \leqslant d} (\lambda_i \lambda_j)^{k_{ij}}$$

of degree $2|\mathbf{k}|$. These notations are needed in order to define the differential operators considered in this paper. We deal with the general case allowing so-called Jacobi weights

$$w_\mu(x_1, \dots, x_d) = (1 - x_1 - \cdots - x_d)^{\mu_0} x_1^{\mu_1} \cdots x_d^{\mu_d} = \boldsymbol{\lambda}^\mu. \tag{1.2}$$

Definition 1. Let $\mu = (\mu_0, \dots, \mu_d) \in \mathbb{R}^{d+1}$ with $\mu_i > -1$ for $0 \leqslant i \leqslant d$. We define

$$U_{\mathbf{k},\mu} := (-1)^\mathbf{k} \frac{1}{\mathbf{k}!} \boldsymbol{\lambda}^{-\mu} \mathbf{D}^\mathbf{k} \Big(\boldsymbol{\lambda}^\mu \Lambda^\mathbf{k}\, \mathbf{D}^\mathbf{k} \Big), \quad \mathbf{k} \in \mathbb{N}_0^R, \tag{1.3}$$

and

$$\mathcal{U}_{\ell,\mu} := \frac{1}{\ell!} \sum_{|\mathbf{k}|=\ell} U_{\mathbf{k},\mu} \quad \text{for} \quad \ell = 0, 1, \ldots \tag{1.4}$$

If $\mu = 0$ we drop the second subscript and write $U_{\mathbf{k}}$ and $\mathcal{U}_{\ell}$ instead. Only formally, for $\mathbf{k} = 0$ and $\ell = 0$, $U_{0,\mu} = \mathcal{U}_{0,\mu}$ is the identity.

Remark 1. The operator $\mathcal{U}_{1,\mu}$ already appears in the work by Derriennic [11], Berens et al. [1], Chen et al. [7] for $\mu = 0$, and in Berens and Xu [2,3], Ditzian [12], Braess and Schwab [4] for general μ. The latter authors denote $\mathcal{U}_{1,\mu}$ as *the Laplacian of the simplex*. Further properties of $U_{\mathbf{k},\mu}$, for $|\mathbf{k}| = 1$, were also studied by Chen and Ditzian [6]. To our knowledge, the operators $\mathcal{U}_{\ell,\mu}$ with $\ell \geqslant 2$ were not considered before, not even in the univariate case.

Remark 2. The powers $\mathcal{U}^{r}_{1,\mu}$ for $r \geqslant 2$ were employed by Derriennic [11], Chen et al. [7] for the definition of the K-functionals

$$K_r(f, t^r)_p := \inf\{\|f - g\|_p + t^r \|\mathcal{U}^{r}_{1} g\|_p;\ g \in C^{2r}(S^d)\}.$$

From a point of view of polynomial approximation, however, these differential operators have the disadvantage that they only annihilate constants. It can easily be seen from definition (1.4) that, in contrast to the powers $\mathcal{U}^{r}_{1,\mu}$, the differential operator $\mathcal{U}_{r,\mu}$ annihilates all polynomials of degree less than r. We show that these operators are more natural for the study of certain quasi-interpolants of Bernstein–Durrmeyer type. For this reason, we prefer the newly defined K-functionals

$$\widetilde{K}_r(f, t^r)_p := \inf\{\|f - g\|_p + t^r \|\mathcal{U}_{r} g\|_p;\ g \in C^{2r}(S^d)\}.$$

The following identities, the first one a pointwise orthogonality statement, and the second one its integrated version, were recently proved in [14].

Theorem 1. *For $n \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{d+1}$, with $|\alpha| = |\beta| = n$, we have*

$$\sum_{|\mathbf{k}| \leqslant n} \frac{(n - |\mathbf{k}|)!}{n!\, \mathbf{k}!} \Lambda^{\mathbf{k}}\, \mathbf{D}^{\mathbf{k}} B_\alpha(\mathbf{x})\, \mathbf{D}^{\mathbf{k}} B_\beta(\mathbf{x}) = \delta_{\alpha,\beta}\, B_\alpha(\mathbf{x}) \tag{1.5}$$

and

$$(n+1)(n+2)\cdots(n+d) \sum_{|\mathbf{k}| \leqslant n} \frac{(n - |\mathbf{k}|)!}{n!\, \mathbf{k}!} \int_{S^d} \Lambda^{\mathbf{k}}\, \mathbf{D}^{\mathbf{k}} B_\alpha(\mathbf{x})\, \mathbf{D}^{\mathbf{k}} B_\beta(\mathbf{x})\, d\mathbf{x} = \delta_{\alpha,\beta}. \tag{1.6}$$

Identity (1.5) is also useful in order to derive an extension of (1.6) including the Jacobi weight (1.2). For this purpose, we define the weighted inner product

$$\langle f, g \rangle_{w_\mu} := \int_{S^d} f(\mathbf{x})\, g(\mathbf{x})\, w_\mu(\mathbf{x})\, d\mathbf{x}\ . \tag{1.7}$$

Corollary 1. *For $n \in \mathbb{N}$, $\mu \in \mathbb{R}^{d+1}$ with $\mu_i > -1$ for all $0 \leqslant i \leqslant d$, let $w_\mu(\mathbf{x}) := \boldsymbol{\lambda}^\mu$ denote the Jacobi weight* (1.2). *Then, for $\alpha, \beta \in \mathbb{N}_0^{d+1}$ with $|\alpha| = |\beta| = n$, we have*

$$\sum_{\ell=0}^{n} \frac{1}{\binom{n}{\ell}} \langle \mathcal{U}_{\ell,\mu} B_\beta \, , \, B_\alpha \rangle_{w_\mu} = \delta_{\alpha,\beta} \, \langle 1, B_\alpha \rangle_{w_\mu} \, . \tag{1.8}$$

Proof. If we multiply identity (1.5) by $\boldsymbol{\lambda}^\mu$ and integrate over S^d, we obtain

$$\delta_{\alpha,\beta} \, \langle 1, B_\alpha \rangle_{w_\mu} = \sum_{|\mathbf{k}| \leqslant n} \frac{(n - |\mathbf{k}|)!}{n! \, \mathbf{k}!} \int_{S^d} \boldsymbol{\lambda}^\mu \, \Lambda^{\mathbf{k}} \, \mathbf{D}^{\mathbf{k}} B_\alpha(\mathbf{x}) \, \mathbf{D}^{\mathbf{k}} B_\beta(\mathbf{x}) \, d\mathbf{x}.$$

Integration by parts yields

$$\begin{aligned}\delta_{\alpha,\beta} \, \langle 1, B_\alpha \rangle_{w_\mu} &= \sum_{|\mathbf{k}| \leqslant n} (-1)^{|\mathbf{k}|} \, \frac{(n - |\mathbf{k}|)!}{n! \, \mathbf{k}!} \int_{S^d} \mathbf{D}^{\mathbf{k}} \Big(\boldsymbol{\lambda}^\mu \, \Lambda^{\mathbf{k}} \, \mathbf{D}^{\mathbf{k}} B_\beta(\mathbf{x}) \Big) \, B_\alpha(\mathbf{x}) \, d\mathbf{x} \\ &= \sum_{|\mathbf{k}| \leqslant n} \frac{(n - |\mathbf{k}|)!}{n!} \int_{S^d} U_{\mathbf{k},\mu} B_\beta(\mathbf{x}) \, B_\alpha(\mathbf{x}) \, w_\mu(\mathbf{x}) \, d\mathbf{x} \\ &= \sum_{\ell=0}^{n} \frac{1}{\binom{n}{\ell}} \, \langle \mathcal{U}_{\ell,\mu} B_\beta \, , \, B_\alpha \rangle_{w_\mu}.\end{aligned}$$

This shows that identity (1.8) is valid. □

It may be noted that the restrictions on the exponents in the Jacobi weight are sufficient in order to have well-defined integrals, and in order to guarantee that integration by parts does not create boundary terms. In the unweighted case, formula (1.8) reduces to (1.6) since $\int_{S^d} B_\alpha(\mathbf{x}) \, d\mathbf{x} = n!/(n+d)!$.

This paper is an extended version of the report [15] with Eqs. (4.4)–(4.7) and Section 5 added.

2. The Bernstein–Durrmeyer operator and quasi-interpolants

The *Bernstein–Durrmeyer operator of order n* was introduced by Durrmeyer [13] and Derriennic [8], and the modified operator with respect to the Jacobi weight w_μ was defined by Berens and Xu [2,3], Ditzian [12]. For $n \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$, and $\mu \in \mathbb{R}^{d+1}$ with $\mu_i > -1$ for all $0 \leqslant i \leqslant d$, the Jacobi-type Bernstein–Durrmeyer operator is given by

$$M_{n,\mu} \; : \; L^p_{w_\mu}(S^d) \to \mathcal{P}_n, \quad f \mapsto \sum_{|\alpha|=n} \frac{\langle f, B_\alpha \rangle_{w_\mu}}{\langle 1, B_\alpha \rangle_{w_\mu}} \, B_\alpha. \tag{2.1}$$

Here the weighted inner product (1.7) is employed, and the domain of the operator is a weighted L^p-space, with $1 \leqslant p \leqslant \infty$, consisting of all measurable functions on S^d with

$$\| f \|_{p,w_\mu} := \Big(\int_{S^d} | f(\mathbf{x})|^p \, w_\mu(\mathbf{x}) \, d\mathbf{x} \Big)^{1/p} < \infty, \quad 1 \leqslant p < \infty$$

and the usual interpretation for $p = \infty$. $\mathcal{P}_n$ denotes the space of algebraic polynomials of total degree at most *n*. This operator is very well understood. Some of its properties are listed by Derriennic [10], see also Ditzian [12], such as

- positivity: $M_{n,\mu} f \geqslant 0$ for every $f \geqslant 0$,
- reproduction of constants: $M_{n,\mu} p = p$ for $p \in \mathcal{P}_0$,
- contractivity: $\|M_{n,\mu} f\|_{p,w_\mu} \leqslant \|f\|_{p,w_\mu}$ for every $f \in L^p_{w_\mu}(S^d)$.

We are going to generalize some of its properties to more general quasi-interpolant operators. The spectral properties of $M_{n,\mu}$ can be described by means of the standard orthogonal decomposition of the Hilbert space $\mathcal{H} := L^2_{w_\mu}(S^d)$ in terms of spaces of orthogonal polynomials,

$$L^2_{w_\mu}(S^d) = \sum_{m=0}^{\infty} \mathcal{E}_{m,\mu}$$
$$\text{with} \quad \mathcal{E}_{0,\mu} := \mathcal{P}_0 \quad \text{and} \quad \mathcal{E}_{m,\mu} := \mathcal{P}_m \cap \mathcal{P}_{m-1}^{\perp} \quad \text{for} \quad m > 0\,.$$

Here, orthogonality refers to the weighted inner product (1.7). It is clear that $M_{n,\mu}$ is a bounded self-adjoint operator on $\mathcal{H}$. The following result by Derriennic [10] (for $\mu = 0$), Berens and Xu [2,3] and Ditzian [12] (for general Jacobi weight) gives a complete characterization of its spectral properties.

Theorem A. *For all $n \in \mathbb{N}_0$, the space $\mathcal{E}_{m,\mu}$, $m \geqslant 0$, is an eigenspace of $M_{n,\mu}$, and $M_{n,\mu} p_m = \gamma_{n,m,\mu}\, p_m$ for all polynomials $p_m \in \mathcal{E}_{m,\mu}$, where*

$$\gamma_{n,m,\mu} := \begin{cases} \dfrac{n!}{(n-m)!}\, \dfrac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+d+|\mu|+m+1)} & \text{for } n \geqslant m, \\ 0 & \text{for } n < m. \end{cases} \tag{2.2}$$

Note that for $f = \sum_{m=0}^{\infty} p_m$, with $p_m \in \mathcal{E}_{m,\mu}$, the operator $M_{n,\mu}$ takes the expression

$$M_{n,\mu}(f) = \sum_{m=0}^{n} \gamma_{n,m,\mu}\, p_m. \tag{2.3}$$

In particular, $M_{n,\mu}$ defines an isomorphism of the space $\mathcal{P}_m$ of all polynomials of degree at most *m*, if $m \leqslant n$. For later reference we give the following expansion:

Lemma 1. *For $n \geqslant m$ we have*

$$\gamma_{n,m,\mu}^{-1} = \sum_{\ell=0}^{m} \frac{\sigma_{\ell,m,\mu}}{n(n-1)\cdots(n-\ell+1)} \tag{2.4}$$

with

$$\sigma_{\ell,m,\mu} := \binom{m}{\ell} \frac{\Gamma(d+m+|\mu|+\ell)}{\Gamma(d+m+|\mu|)} = \binom{m}{\ell} \prod_{j=0}^{\ell-1} (j+d+m+|\mu|)\,. \tag{2.5}$$

Proof. Multiplying Eq. (2.4) by $n!/(n-m)!$ leads to the equivalent identity

$$\prod_{i=1}^{m}(n+d+|\mu|+i) = \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\Gamma(d+m+|\mu|+\ell)}{\Gamma(d+m+|\mu|)}\prod_{j=\ell}^{m-1}(n-j)\ .$$

This identity follows from the polynomial interpolation formula for

$$\begin{aligned}\omega_{m,\mu}(x) &:= \prod_{i=1}^{m}(x+d+|\mu|+i) = \frac{\Gamma(x+d+|\mu|+m+1)}{\Gamma(x+d+|\mu|+1)} \\ &= \sum_{\ell=0}^{m}[y_0,\ldots,y_\ell \mid \omega_{m,\mu}]\prod_{i=0}^{\ell-1}(x-y_i)\end{aligned}$$

based on the divided differences with respect to the nodes $y_i = m-1-i$, $i=0,\ldots,m$. By straightforward calculation of the Newton scheme we find

$$[y_0,\ldots,y_\ell \mid \omega_{m,\mu}] = \binom{m}{\ell}\frac{\Gamma(d+2m+|\mu|-\ell)}{\Gamma(d+m+|\mu|)} = \sigma_{m-\ell,m,\mu},$$
$$\ell = 0,\ldots,m. \qquad \square$$

Next we show that the combination of Theorem 1 and Theorem A provides an explicit formulation for the inverse of $M_{n,\mu}|_{\mathcal{P}_n}$ as the restriction of a bounded self-adjoint operator on $\mathcal{H}$ which maps $\mathcal{H}$ onto $\mathcal{P}_n$. This operator can then be used in order to define quasi-interpolants on $\mathcal{H}$ or $L^p_{w_\mu}(S^d)$. As a first step in this direction, we prove the following

Lemma 2. *The differential operators $U_{\mathbf{k},\mu}$, $\mathbf{k} \in \mathbb{N}_0^R$, and $\mathcal{U}_{\ell,\mu}$, $\ell \geqslant 0$, are densely defined symmetric operators on the Hilbert space $\mathcal{H}$. They commute with the Bernstein–Durrmeyer operator $M_{n,\mu}$, $n \in \mathbb{N}_0$.*

Proof. The operator $U_{\mathbf{k},\mu}$ is defined for all $f \in C^\infty(S^d)$ which is a dense subspace of $\mathcal{H}$. Integration by parts shows that

$$\langle U_{\mathbf{k},\mu}f, g\rangle_{w_\mu} = \langle f, U_{\mathbf{k},\mu}g\rangle_{w_\mu}, \qquad f,g \in C^\infty(S^d),$$

which proves that $U_{\mathbf{k},\mu}$ is symmetric. The same properties are valid for the operator $\mathcal{U}_{\ell,\mu}$.

It is clear from definition (1.3) that $U_{\mathbf{k},\mu}$ maps the space $\mathcal{P}_m$ into itself. Hence, for $p \in \mathcal{E}_{m,\mu}$ and $q \in \mathcal{P}_{m-1}$, we find

$$\langle U_{\mathbf{k},\mu}p,\ q\rangle_{w_\mu} = \langle p,\ U_{\mathbf{k},\mu}q\rangle_{w_\mu} = 0.$$

This shows that $U_{\mathbf{k},\mu}$ maps $\mathcal{E}_{m,\mu}$ into itself as well. Therefore, we conclude from Theorem A that $U_{\mathbf{k},\mu}$ commutes with $M_{n,\mu}$. $\square$

Theorem 1 allows us to describe the inverse of the Bernstein–Durrmeyer operator $M_{n,\mu}$, restricted to $\mathcal{P}_n$, in terms of the differential operators $\mathcal{U}_{\ell,\mu}$, $0 \leqslant \ell \leqslant n$.

Theorem 2. *For any $n \in \mathbb{N}$ let*

$$Y_{n,\mu} : C^{2n}(S^d) \to \mathcal{P}_n, \quad f \mapsto \sum_{\ell=0}^{n} \frac{1}{\binom{n}{\ell}} \mathcal{U}_{\ell,\mu} f. \tag{2.6}$$

Then $Y_{n,\mu}$ is a symmetric operator, and for all $p \in \mathcal{P}_n$, we have

$$M_{n,\mu} Y_{n,\mu} p = Y_{n,\mu} M_{n,\mu} p = p. \tag{2.7}$$

Proof. By Lemma 2, $Y_{n,\mu}$ is symmetric. In order to prove identity (2.7), we expand p in terms of the Bernstein polynomial basis. As a consequence of Corollary 1, we obtain

$$p = \sum_{\ell=0}^{n} \frac{1}{\binom{n}{\ell}} M_{n,\mu}(\mathcal{U}_{\ell,\mu} p),$$

which gives $M_{n,\mu} Y_{n,\mu} p = p$. Lemma 2 also shows that $Y_{n,\mu} M_{n,\mu} p = p$. □

In the previous paper [14], the last two authors have introduced quasi-interpolants which are between the Bernstein–Durrmeyer operator $M_n := M_{n,0}$ and the orthogonal projector onto $\mathcal{P}_n$. By employing the differential operators again and by introducing the Jacobi weight into these operators, we can define the *Bernstein–Durrmeyer quasi-interpolant of order* (r, n), $0 \leqslant r \leqslant n$, with Jacobi weight w_μ, by

$$M_{n,\mu}^{(r)} : \ L^p_{w_\mu}(S^d) \to \mathcal{P}_n, \qquad f \mapsto \sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \mathcal{U}_{\ell,\mu}(M_{n,\mu} f). \tag{2.8}$$

Apparently, $M_{n,\mu}^{(r)}$ is a bounded linear operator, which is self-adjoint if $p = 2$. Moreover, $M_{n,\mu}^{(0)} = M_{n,\mu}$, while $M_{n,\mu}^{(n)}$ is the orthogonal projection onto $\mathcal{P}_n$. Theorem 2 recovers the statement (already proved in [14] for the unweighted case) that $M_{n,\mu}^{(r)}$ *reproduces all polynomials from* $\mathcal{P}_r$.

Remark 3. Different operators of Bernstein–Durrmeyer type of order r were constructed by Derriennic [11] for the case $\mu = 0$. These operators have the form

$$Q_n^{(r)} f := M_n f + \sum_{\ell=1}^{r} \alpha_{n,\ell} \, \mathcal{U}_1^{\ell}(M_n f),$$

where $\alpha_{n,1} := (\frac{1}{n+1} + \cdots + \frac{1}{n+d})/d$ and $\alpha_{n,\ell}$ is given by a complicated recursive definition. Note that the powers of the operator $\mathcal{U}_1$ appear in this definition. Moreover, for $r = 1$ already, the operator $Q_n^{(1)}$ does not reproduce linear polynomials. Hence, our quasi-interpolants $M_{n,\mu}^{(r)}$ are different from these operators. They are also different to Sablonnière's quasi-interpolants [16,17].

3. A Voronovskaja-type result

The following result of Voronovskaja type enlightens the role that is played by the differential operator $\mathcal{U}_{1,\mu}$. The result for $\mu = 0$ appeared in Derriennic [8] for $d = 1$ and in Derriennic [9] for $d > 1$. The case of general μ was given in Berens and Xu [2] ($d = 1$) and Ditzian [12, Remark 4.2] ($d > 1$).

Theorem B. *For all $n \in \mathbb{N}_0$ and $f \in C^2(S^d)$ we have*

$$\lim_{n\to\infty} n \left(f(x) - M_{n,\mu} f(x)\right) = \mathcal{U}_{1,\mu} f(x). \tag{3.1}$$

Remark 4. Ditzian uses the factor $v_{n,\mu} := \sum_{k=n+1}^{\infty} (k(k+d+|\mu|))^{-1}$ instead of n. It is easy to prove that $\lim_{n\to\infty} v_{n,\mu}/n = 1$, and therefore identity (3.1) is an equivalent formulation of his result.

The following extension of Theorem B refers to our quasi-interpolants $M_{n,\mu}^{(r)}$ in (2.8). This result is new even for $d = 1$ and $\mu = 0$.

Theorem 3. *For $r \in \mathbb{N}_0$ and $f \in C^{2(r+1)}(S^d)$, we have*

$$\lim_{n\to\infty} \binom{n}{r+1} \left(f - M_{n,\mu}^{(r)} f\right)(\mathbf{x}) = \mathcal{U}_{r+1,\mu} f(\mathbf{x}), \quad \mathbf{x} \in S^d,$$

and the convergence is uniform with respect to $\mathbf{x}$.

It is easy to prove the theorem for polynomials. Since this result will be employed in the proof of Theorem 4 below, we state it separately. The full proof of Theorem 3 will be given in Section 5.

Lemma 3. *Theorem* 3 *holds true for polynomials f.*

Proof. It is sufficient to show the result for any basis of the polynomial space. Let $f = p_m \in \mathcal{E}_{m,\mu}$. In case $m \leqslant r \leqslant n$ we have $p_m = M_{n,\mu}^{(r)} p_m$ and $\mathcal{U}_{r+1,\mu} p_m = 0$, so that the statement holds trivially. In case $r < m \leqslant n$ we have, by Theorem 2 and Eq. (2.8),

$$\gamma_{n,m,\mu}^{-1} \left(p_m - M_{n,\mu}^{(r)} p_m\right) = \sum_{\ell=r+1}^{m} \frac{1}{\binom{n}{\ell}} \mathcal{U}_{\ell,\mu} p_m,$$

whence

$$\gamma_{n,m,\mu}^{-1} \binom{n}{r+1} \left(p_m - M_{n,\mu}^{(r)} p_m\right) = \mathcal{U}_{r+1,\mu} p_m + \sum_{\ell=r+2}^{m} \frac{\binom{n}{r+1}}{\binom{n}{\ell}} \mathcal{U}_{\ell,\mu} p_m .$$

Since

$$\lim_{n\to\infty} \gamma_{n,m,\mu} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\binom{n}{r+1}}{\binom{n}{\ell}} = 0 \quad \text{for} \quad \ell = r+2, \ldots, m,$$

our result follows. $\square$

4. Spectral analysis

We have shown in Lemma 2 that the spaces $\mathcal{E}_{m,\mu}$ are invariant subspaces of the differential operators $U_{\mathbf{k},\mu}$ and $\mathcal{U}_{\ell,\mu}$, respectively, for any $\mathbf{k} \in \mathbb{N}_0^R$ and $\ell \in \mathbb{N}_0$. We now verify that these are also eigenspaces of all operators $\mathcal{U}_{\ell,\mu}$, thus extending the result

$$\mathcal{U}_{1,\mu}\, p_m = m(m+d+|\mu|)\, p_m, \qquad p_m \in \mathcal{E}_{m,\mu}, \tag{4.1}$$

which for $\mu = 0$ is due to Derriennic [11] and Berens et al. [1], and for general μ due to Berens and Xu [2] (for $d = 1$), and Ditzian [12] (for $d > 1$).

Theorem 4. *For all $r, m \in \mathbb{N}_0$ and $p_m \in \mathcal{E}_{m,\mu}$ we have*

$$\mathcal{U}_{r,\mu}\, p_m = \frac{\sigma_{r,m,\mu}}{r!}\, p_m \tag{4.2}$$

with $\sigma_{r,m,\mu}$ as defined in Lemma 1. In particular, $\sigma_{r,m,\mu} = 0$ for $r > m$.

Proof. We use induction on r and employ an idea used in the proof of (4.1) in Chen and Ditzian [5]. The result is trivial for $r = 0$ and for $0 \leqslant m < r$. Let us assume that the result is true for all $0 \leqslant i \leqslant r-1$ and let $m \in \mathbb{N}_0$, $r \leqslant m$, and $p_m \in \mathcal{E}_{m,\mu}$.

The Voronovskaja result of Lemma 3 yields

$$\lim_{n\to\infty} \binom{n}{r} \left(p_m - M_{n,\mu}^{(r-1)} p_m\right) = \mathcal{U}_{r,\mu} p_m.$$

We insert (2.8) and use the induction hypothesis to obtain

$$\begin{aligned} \mathcal{U}_{r,\mu} p_m &= \lim_{n\to\infty} \binom{n}{r} \left(p_m - \sum_{\ell=0}^{r-1} \frac{1}{\binom{n}{\ell}}\, \mathcal{U}_{\ell,\mu}(M_{n,\mu} p_m)\right) \\ &= \lim_{n\to\infty} \binom{n}{r} \left(1 - \gamma_{n,m,\mu} \sum_{\ell=0}^{r-1} \frac{1}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!}\right) p_m. \end{aligned}$$

By Lemma 1 we find

$$\mathcal{U}_{r,\mu} p_m = \lim_{n\to\infty} \gamma_{n,m,\mu} \sum_{\ell=r}^{m} \frac{\binom{n}{r}}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!}\, p_m = \frac{\sigma_{r,m,\mu}}{r!}\, p_m.$$

This completes the induction. $\square$

Remark 5. For $\mu = 0$, we obtain from Theorem 4 that

$$\mathcal{U}_r p_m = \binom{m}{r} \binom{d+m+r-1}{r} p_m\,, \qquad p_m \in \mathcal{E}_m, \tag{4.3}$$

which is a more compact formula for the eigenvalues of $\mathcal{U}_r$.

Remark 6. The following identities for the differential operators $\mathcal{U}_{r,\mu}$ follow from Theorem 4. For all $r \geqslant 0$ we have the recurrence relation

$$\mathcal{U}_{r+1,\mu} f = \frac{1}{(r+1)^2}[\mathcal{U}_{1,\mu} - r(r+d+|\mu|)\, I\,]\mathcal{U}_{r,\mu} f, \quad f \in C^{2r+2}(S^d), \tag{4.4}$$

and the product formula

$$\mathcal{U}_{r,\mu} f = \frac{1}{(r!)^2} \prod_{m=0}^{r-1} (\mathcal{U}_{1,\mu} - m(m+d+|\mu|)I) f, \quad f \in C^{2r}(S^d). \tag{4.5}$$

The proof of (4.5) for $\mu = 0$ was first communicated to us by M. Felten. The analogous argument for general μ is as follows. Simple calculations show that, for $m \geqslant 0$, we have

$$\begin{aligned} \sigma_{0,m,\mu} &= 1 \quad \text{and} \\ \sigma_{r+1,m,\mu} &= \frac{(m-r)(m+d+|\mu|+r)}{r+1}\, \sigma_{r,m,\mu} \\ &= \frac{m(m+d+|\mu|) - r(r+d+|\mu|)}{r+1}\, \sigma_{r,m,\mu}, \quad 0 \leqslant r < m. \end{aligned} \tag{4.6}$$

This yields (4.4), via (4.2) and (4.1), first for $f = p_m \in \mathcal{E}_{m,\mu}$, and hence for all polynomials. The identity (4.4) follows for all $f \in C^{2r+2}(S^d)$ by a density argument, and (4.5) is obtained by induction.

For later use we also describe the spectral properties of the quasi-interpolants $M_{n,\mu}^{(r)}$. The following statement follows from Theorem A, Theorem 4, Lemma 1 and the fact that the operators $M_{n,\mu}$ and $\mathcal{U}_{l,\mu}$ commute.

Lemma 4. *For all $n, m, r \in \mathbb{N}_0$, $0 \leqslant r \leqslant n$, the spaces $\mathcal{E}_{m,\mu}$ are eigenspaces of the operator $M_{n,\mu}^{(r)}$. Namely, for $p_m \in \mathcal{E}_{m,\mu}$ we have*

$$M_{n,\mu}^{(r)}\, p_m = \lambda_{n,m,\mu}^{(r)}\, p_m$$

with the eigenvalues

$$\begin{aligned} \lambda_{n,m,\mu}^{(r)} &= \gamma_{n,m,\mu} \sum_{\ell=0}^{r} \frac{1}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!} \\ &= \begin{cases} 0 & if\ m > n, \\ 1 & if\ n \geqslant r \geqslant m, \\ 1 - \gamma_{n,m,\mu} \displaystyle\sum_{\ell=r+1}^{m} \frac{1}{\binom{n}{\ell}} \frac{\sigma_{\ell,m,\mu}}{\ell!} & if\ r < m \leqslant n. \end{cases} \end{aligned} \tag{4.7}$$

5. Approximation order

In this section we study approximation properties of the quasi-interpolants $M_{n,\mu}^{(r)}$, $0 \leqslant r \leqslant n$. Our first basic result is

Lemma 5. *Let $f \in C^{2r}(S^d)$. Then $M_{n,\mu}^{(r)} f \to f$ as $n \to \infty$, uniformly on S^d.*

Proof. Using (2.8) and Lemma 2, we have

$$M_{n,\mu}^{(r)} f = M_{n,\mu} f + \sum_{\ell=1}^{r} \frac{1}{\binom{n}{\ell}} M_{n,\mu}(\mathcal{U}_{\ell,\mu} f), \quad f \in C^{2r}(S^d),$$

where $M_{n,\mu}$ is the usual Bernstein–Durrmeyer operator (2.1). This representation and the contractivity property of $M_{n,\mu}$ yield the estimate

$$\|M_{n,\mu}^{(r)} f - f\| \leqslant \|M_{n,\mu} f - f\| + \sum_{\ell=1}^{r} \frac{1}{\binom{n}{\ell}} \|\mathcal{U}_{\ell,\mu} f\|,$$

with $\|\cdot\| = \|\cdot\|_\infty$ the maximum norm on S^d. It is well known that $\|M_{n,\mu} f - f\| \to 0$ as $n \to \infty$ for all continuous functions f. Since $\|\mathcal{U}_{\ell,\mu} f\|$, $\ell = 1, \ldots, r$, are bounded, the result follows. $\square$

Before we discuss the approximation order for our quasi-interpolants, we establish the following useful relation for the eigenvalues $\lambda_{n,m,\mu}^{(r)}$. For $r = 0$, this relation was proved in Derriennic [11] and Berens et al. [1] (for $\mu = 0$), Berens and Xu [2] (for $d = 1$), and Ditzian [12]. For $r > 0$, this relation is new even in the one-dimensional case.

Lemma 6. *For $n, m, r \in \mathbb{N}_0$, $0 \leqslant r \leqslant n$, the following difference equation holds true*:

$$\lambda_{n,m,\mu}^{(r)} - \lambda_{n-1,m,\mu}^{(r)} = \frac{r+1}{n+d+|\mu|} \frac{1}{\binom{n}{r+1}} \frac{\sigma_{r+1,m,\mu}}{(r+1)!} \gamma_{n,m,\mu}\,. \tag{5.1}$$

Proof. From (4.7), (2.2) and by simple calculations, we obtain

$$\begin{aligned}
\lambda_{n,m,\mu}^{(r)} - \lambda_{n-1,m,\mu}^{(r)} &= \sum_{\ell=0}^{r} \frac{\sigma_{\ell,m,\mu}}{\ell!} \left[\frac{\gamma_{n,m,\mu}}{\binom{n}{\ell}} - \frac{\gamma_{n-1,m,\mu}}{\binom{n-1}{\ell}} \right] \\
&= \sum_{\ell=0}^{r} \frac{\sigma_{\ell,m,\mu}}{\ell!} \left[1 - \frac{(n-m)(n+d+m+|\mu|)}{(n-\ell)(n+d+|\mu|)} \right] \frac{\gamma_{n,m,\mu}}{\binom{n}{\ell}} \\
&= \sum_{\ell=0}^{r} \frac{\sigma_{\ell,m,\mu}}{\ell!} \left[\frac{(m-\ell)(m+d+|\mu|+\ell) - (n-\ell)\ell}{(n-\ell)(n+d+|\mu|)} \right] \frac{\gamma_{n,m,\mu}}{\binom{n}{\ell}}.
\end{aligned}$$

Eq. (4.6) yields

$$\lambda_{n,m,\mu}^{(r)} - \lambda_{n-1,m,\mu}^{(r)} = \frac{\gamma_{n,m,\mu}}{n+d+|\mu|} \sum_{\ell=0}^{r} \left[\frac{(\ell+1)\sigma_{\ell+1,m,\mu}}{(\ell+1)!\binom{n}{\ell+1}} - \frac{\ell \sigma_{\ell,m,\mu}}{\ell! \binom{n}{\ell}} \right],$$

and an obvious cancellation gives (5.1). □

This lemma is the basis for a telescoping argument in order to find an error formula for the quasi-interpolant. Eq. (5.1) implies

$$M_{n,\mu}^{(r)} f - M_{n-1,\mu}^{(r)} f = \frac{r+1}{n+d+|\mu|} \binom{n}{r+1}^{-1} \mathcal{U}_{r+1,\mu} M_{n,\mu} f \tag{5.2}$$

for $f \in \mathcal{E}_m$ and, by density, for all $f \in L^p_{w_\mu}(S^d)$. Note that the operators $M_{n,\mu}^{(r)}$, $M_{n-1,\mu}^{(r)}$ and $M_{n,\mu}$ map $L^p_{w_\mu}(S^d)$ onto $\mathcal{P}_n$. Since $\mathcal{P}_n$ is finite dimensional, all operators in (5.2) are bounded on $L^p_{w_\mu}(S^d)$. Moreover, for smooth functions we may interchange the differential operator and the Bernstein–Durrmeyer operator again. This leads to our main result in this section.

Theorem 5. *If $f \in C^{2r+2}(S^d)$, then*

$$f - M_{n,\mu}^{(r)} f = \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|} \binom{\ell}{r+1}^{-1} M_{\ell,\mu}(\mathcal{U}_{r+1,\mu} f), \quad f \in C^{2r+2}(S^d), \tag{5.3}$$

where the infinite series converges absolutely and uniformly on S^d. Consequently, for $1 \leqslant p \leqslant \infty$, we have

$$\| f - M_{n,\mu}^{(r)} f \|_p \leqslant \frac{C_{r,d,n,\mu}}{\binom{n}{r+1}} \| \mathcal{U}_{r+1,\mu} f \|_p, \tag{5.4}$$

where

$$C_{r,d,n,\mu} := \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|} \frac{\binom{n}{r+1}}{\binom{\ell}{r+1}} \to 1, \quad n \to \infty, \tag{5.5}$$

for fixed r, d and μ.

Proof. Identity (5.3) follows from (5.2) and Lemma 5 by writing the left-hand side of (5.3) as a telescoping series. By making use of the triangle inequality and the contractivity of $M_{n,\mu}$, we obtain the absolute and uniform convergence of the series in (5.3), and also inequality (5.4), where the constants $C_{r,d,n,\mu}$ are defined as in (5.5). For the reader's convenience we verify the asymptotic behavior of these constants. For this purpose, we introduce the notations

$$\underline{\beta}_{d,n,\mu} := \min\left\{1, \frac{n+2}{n+1+d+|\mu|}\right\},$$
$$\overline{\beta}_{d,n,\mu} := \max\left\{1, \frac{n+2}{n+1+d+|\mu|}\right\}.$$

We readily obtain that

$$\underline{\beta}_{d,n,\mu} \leqslant \frac{\ell+1}{\ell+d+|\mu|} \leqslant \overline{\beta}_{d,n,\mu}, \quad \ell \geqslant n+1,$$

and this gives

$$\frac{C_{r,d,n,\mu}}{(r+1)!\binom{n}{r+1}} = \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|} \prod_{j=0}^{r} \frac{1}{\ell-j}$$
$$\left\{ \begin{array}{l} \leqslant \overline{\beta}_{d,n,\mu} \\ \geqslant \underline{\beta}_{d,n,\mu} \end{array} \right\} \sum_{\ell=n+1}^{\infty} (r+1) \prod_{j=0}^{r+1} \frac{1}{\ell+1-j}.$$

The last series equals

$$\sum_{\ell=n+1}^{\infty} \left[\prod_{j=0}^{r} \frac{1}{\ell-j} - \prod_{j=0}^{r} \frac{1}{\ell+1-j} \right] = \prod_{j=0}^{r} \frac{1}{n+1-j} = \frac{1}{(r+1)!\binom{n+1}{r+1}}.$$

The result $\lim_{n\to\infty} C_{r,d,n,\mu} = 1$ follows directly. □

Inequality (5.4) is a so-called direct theorem of Jackson–Favard-type. Based on (5.3), we are now able to complete the proof of Theorem 3.

Proof of Theorem 3. Identity (5.3) implies

$$\begin{aligned} &\binom{n}{r+1}\left(f - M_{n,\mu}^{(r)} f\right) - \mathcal{U}_{r+1,\mu} f \\ &\quad = \left(C_{r,d,n,\mu}\mathcal{U}_{r+1,\mu} f - \mathcal{U}_{r+1,\mu} f\right) \\ &+ \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|} \frac{\binom{n}{r+1}}{\binom{\ell}{r+1}} \left(M_{\ell,\mu}(\mathcal{U}_{r+1,\mu} f) - \mathcal{U}_{r+1,\mu} f\right) \end{aligned}$$

with the constant $C_{r,d,n,\mu}$ as in (5.5). The first term on the right-hand side converges to zero as $n \to \infty$, by (5.5), and the second term converges to zero as well, since

$$\begin{aligned} &\left\| \sum_{\ell=n+1}^{\infty} \frac{r+1}{\ell+d+|\mu|} \frac{\binom{n}{r+1}}{\binom{\ell}{r+1}} \left(M_{\ell,\mu}(\mathcal{U}_{r+1,\mu} f) - \mathcal{U}_{r+1,\mu} f\right) \right\| \\ &\quad \leqslant C_{r,d,n,\mu} \sup_{\ell > n} \| M_{\ell,\mu}(\mathcal{U}_{r+1,\mu} f) - \mathcal{U}_{r+1,\mu} f \| \end{aligned}$$

and

$$\lim_{n\to\infty} \| M_{n,\mu}(\mathcal{U}_{r+1,\mu} f) - \mathcal{U}_{r+1,\mu} f \|_\infty = 0, \quad f \in C^{2r+2}(S^d).$$

This completes the proof of Theorem 3 for smooth *f*. □

We conclude the paper with a generalization of the direct and inverse theorem by Braess and Schwab [4, Theorem 3.3] for the Hilbert space $\mathcal{H} = L^2_{w_\mu}(S^d)$. This is an easier argument, and is based on Theorem 4 alone. Here we make use of the following Sobolev-type semi-norm $|f|_{\ell,\mu}$ of integer order $\ell \geqslant 0$, where

$$|f|^2_{\ell,\mu} := \sum_{m=0}^{\infty} \langle \mathcal{U}_{\ell,\mu} p_m, p_m \rangle_{w_\mu} = \frac{1}{\ell!} \sum_{m=0}^{\infty} \sigma_{\ell,m,\mu} \langle p_m, p_m \rangle_{w_\mu}$$

and $f = \sum_{m=0}^{\infty} p_m$ is the orthogonal decomposition as in Theorem A. The corresponding smoothness spaces are given by

$$\mathcal{H}^k_\mu := \{ f \in L^2_{w_\mu}(S^d) \mid |f|_{\ell,\mu} < \infty \text{ for } \ell = 0, \ldots, k \}, \qquad k \in \mathbb{N}.$$

Theorem 6. *Let $k \geqslant \ell \geqslant 0$ be integers.*

(a) *Direct result*: *For $f \in \mathcal{H}^k_\mu$ and $n \in \mathbb{N}_0$, we have*

$$\inf_{p \in \mathcal{P}_n} |f - p|^2_{\ell,\mu} \quad \leqslant \quad \frac{k!}{\ell!} c_{n,k,\ell} \, |f|^2_{k,\mu},$$

where $c_{n,k,\ell} = \sup_{m \geqslant n+1} \frac{\sigma_{\ell,m,\mu}}{\sigma_{k,m,\mu}} = \frac{\sigma_{\ell,n+1,\mu}}{\sigma_{k,n+1,\mu}}$.

(b) *Converse result*: *For $n \in \mathbb{N}_0$ and $p \in \mathcal{P}_n$, we have*

$$|p|^2_{k,\mu} \quad \leqslant \quad \frac{\ell!}{k!} d_{n,k,\ell} \, |p|^2_{\ell,\mu},$$

where $d_{n,k,\ell} = \max_{m=0,\ldots,n} \frac{\sigma_{k,m,\mu}}{\sigma_{\ell,m,\mu}} = \frac{\sigma_{k,n,\mu}}{\sigma_{\ell,n,\mu}}$.

Apparently, by inspection of (2.5), we obtain

$$c_{n,k,\ell} = \mathcal{O}\,(n^{-2(k-\ell)}) \qquad \text{and} \qquad d_{n,k,\ell} = \mathcal{O}\,(n^{2(k-\ell)}) \, .$$

Acknowledgments

We thank Michael Felten for pointing out to us that the product representation (4.5) of the differential operator $\mathcal{U}_{r,\mu}$ is valid, after we communicated our Theorem 4 to him. This hint helped us in our approach to the Jackson–Favard result of Theorem 5.

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